QMC integration of improper integrals

An overview with non-uniform sequences in mind

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Contents

When do we get singular integrands?

- In finance: expectation values / integrals over $\mathbb{R}^s$, transformed to $[0,1]^s$ using inversion
- E.g. Payoff of Asian call option with log-return $x^{(i)} \sim H_i$:

$$I = \int_{\mathbb{R}^m} \left(\frac{S_0}{m} \sum_{i=1}^{m} e^{\sum_{j=1}^{i} x^{(j)}} - K \right)^+ dH_1(x^{(1)}) \dots dH_m(x^{(m)})$$

$$= \int_{[0,1]^m} \left(\frac{S_0}{m} \sum_{i=1}^{m} e^{\sum_{j=1}^{i} H_j^{-1}(1-x^{(j)})} - K \right)^+ dx^{(1)} \dots dx^{(m)}$$

Singularity on the whole lower boundary of $[0,1]^m$

Sobol's theorem, uniform distribution

Theorem (Sobol): Let $c_N = \min_{1\le n\le N, 1\le s\le m} x_n^{(s)}$, and $G_{i'}(\epsilon) = \left\{\mathbf{x} \in K : \prod_{i\in i'} x^{(i)} \ge \epsilon\right\}$ for $i' \subseteq \{1, 2, \ldots, m\}$. If $\int_{K_{i'}} x_{i_1} \ldots c_{i_s} \left| f^{(i')}(\mathbf{x})\right| dx_{i_1} \ldots dx_{i_s}$ converge for all i' and for $n \to \infty$ we have

$$D_N^{(i')} \int_{G_{i'}(c_N)} \left| f^{(i')}(\mathbf{x})\right| dx_{i_1} \ldots dx_{i_s} = o(1),$$

then the QMC sum converges to the actual value of the integral.

Owen (2004) gave explicit error estimates under certain growth conditions using a low-variance extension $\hat{f}$ of f:

$$\left|\hat{I} - I\right| = \mathcal{O}\left(n^{-1+\epsilon+r\max_j A_j}\right)$$

non-uniform singular integration

Theorem (Hartinger, K., Tichy): $f(\mathbf{x}) : [\mathbf{a}, \mathbf{b}] \mapsto \mathbb{R}$, singular at left boundary, H prob. dist. $\omega = (\mathbf{y}_1, \mathbf{y}_2, \ldots) \subset [\mathbf{a}, \mathbf{b}]$. Let $a^{(j)} < c^{(j)} \leq c_N^{(j)} = \min_{1 \leq n \leq N} y_n^{(j)}$. If the integral exists and

$$D_{N,H}(\omega)\, V_{[\mathbf{c},\mathbf{b}]}(f) = o(1),$$

$\Rightarrow$ QMC estimator converges to value of the improper integral:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\mathbf{y}_n) = \int_{[\mathbf{a},\mathbf{b}]} f(\mathbf{x}) dH(\mathbf{x}) .$$

- Theorem uses L-shaped region (Owen's $K_{\text{min}}^{\text{orig}}$), Sobol used hyperbolic region (Owen's $K_{\text{prod}}^{\text{orig}}$).
- Cut-Off c_N depends on how fast the sequence tends to the origin.

Obtaining error orders

Theorem (Hartinger, K., Predota): For $H(\mathbf{x}) = H_1(x_1)...H_m(x_m)$ with $h_i(x) < h_\lambda(x)$ for all $x > x_0$ and some λ and x_0 ($h_\lambda(x)$... double-exp. dist.) and $1 \le i \le m$, the convergence order for the Asian option valuation using digital $(0, s)$ or Halton sequences and the inversion method is

$$\mathcal{O}\left(\frac{\log^m N}{N^{1-m/\lambda}}\right).$$

Sketch of Proof: Bound easy for double exp. distr. (log and exp cancel each other), then bound all lighter-tailed distributions by this.

- all $(0, s)$ sequences have the same bound

Non-uniformly distr. seqs.: Hlawka-Mück

Theorem (Hlawka, 1997): Let $H(\mathbf{x}) = H_1(x^{(1)}) \cdot \ldots \cdot H_m(x^{(m)})$ a DF $[0,1]^m$ and $M_h = \sup h(\mathbf{x})$. Let furthermore $\omega = (\mathbf{x_1}, \ldots, \mathbf{x_N})$ be a sequence in $[0,1]^m$. Then the point set $\omega = (\mathbf{y_1}, \ldots, \mathbf{y_N})$ with

$$y_k^{(j)} = \frac{1}{N} \sum_{r=1}^{N} \left\lfloor 1 + x_k^{(j)} - H_j\left(x_r^{(j)}\right) \right\rfloor = \frac{1}{N} \sum_{r=1}^{N} \chi_{[0, x_k^{(j)}]}\left(H_j\left(x_r^{(j)}\right)\right)$$

has an H-discrepancy of

$$D_{N,H}(\tilde{\omega}) \le (1 + 4M_h)^m D_N(\omega) .$$

- Generated on lattice $\Rightarrow$ some identical points
- Points might be generated at 0
- If $H(\mathrm{x})$ does not factor $\Rightarrow$ discrepancy not of $\mathcal{O}(\log^m N/N)$.

How to make the sequences avoid the origin

- Uniformly distributed low-discrepancy sequences already avoid the origin with order $\mathcal{O}(\frac{1}{N})$ or $\mathcal{O}(\frac{1}{N^2})$.
- Transformed sequences do not necessarily have this property! Solutions:
  - Move all elements $\leq \frac{1}{N}$ to $\frac{1}{N}$ $\Rightarrow$ only small number of points needs to be modified, discrepancy bound stays the same (Hartinger, K., Tichy)
  - Scale $[0,1]^m$ to $[\epsilon_N, 1-\epsilon_N]^m$ $\Rightarrow$ all points modified, relative distances are preserved, same discrepancy bound (Owen)

Obtaining error orders for H-M.

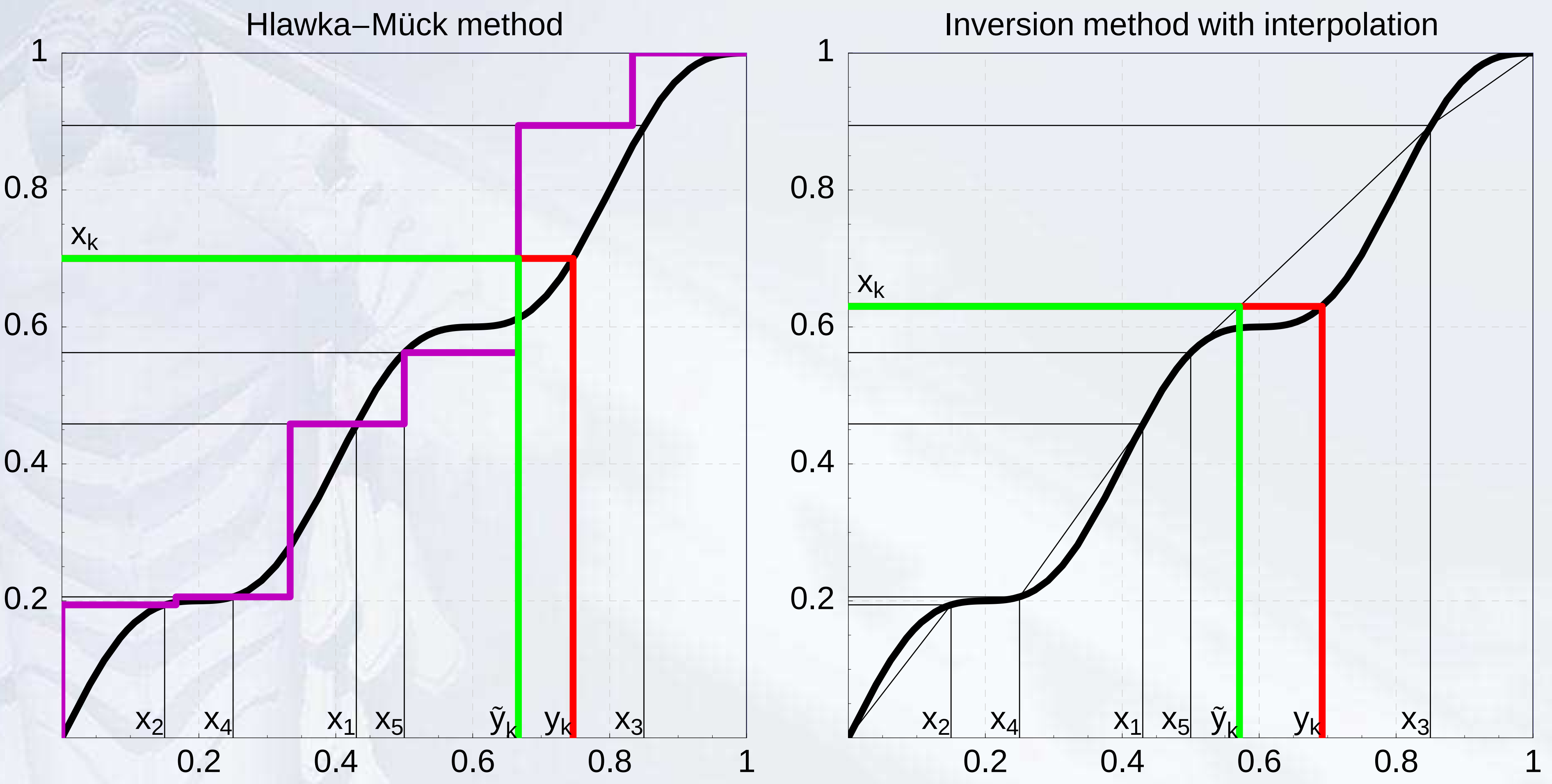
Theorem (Hartinger, K., Predota): Using the Hlawka-Mück-type method for a H-distributed sequence $\bar{\omega}$, the convergence order of the direct QMC algorithm for the improper integration problem of the Asian call option can be bounded by

$$\mathcal{O}\left(\frac{\log^m N}{N^{1-m/\lambda}}\right).$$

Here λ denotes the parameter of the double-exponential distribution used to transform from $\mathbb{R}^m$ to the interval $[0,1]^m$ and back.

Sketch of Proof: Direct estimation of all terms in the proof of the general convergence theorem.

How the Hlawka-Mück method works



- Zwei Grafiken, eine mit HM, eine mit Interpolation

Decoupling approximation and inversion

Idea: Approximate DF with sequence $\hat{\omega}$, use inversion of seq. ω.
Advantage: If $H(\mathrm{x}) = H^{(1)}(x_1) \cdot \ldots \cdot H^{(1)}(x_m)$, then $H^{(1)}$ does not have to be calculated for each dimension separately

- If not $f(x) = 0$ a.s. (in ball around 0) $\Rightarrow$ no points generated at 0
- Lower bound for distance to 0 can be given using the derivatives of H $\Rightarrow$ transformed sequence also avoids the origin (no move necessary), depending on the used distribution.

Convergence theorem

Theorem (Hartinger, K.): Let $\hat{\omega} = (z_i)_{i \leq N} \subset [0,1]$ the support sequence, $\omega = (\mathbf{x}_i)_{i \leq N} \subset [0,1]^m$ an m-dim. sequence. Define

$$\mathbf{z}_k^{(l)-} = \max_{\left\{z_i \in \hat{\omega} \,:\, H_l(z_i) \leq \mathbf{x}_k^{(l)}\right\}} z_i \quad \text{and} \quad \mathbf{z}_k^{(l)+} = \min_{\left\{z_i \in \hat{\omega} \,:\, H_l(z_i) \geq \mathbf{x}_k^{(l)}\right\}} z_i .$$

Then the H-discrepancy of any transformed sequence $\bar{\omega} = (y_k)_{1 \leq k \leq N}$ with $y_k^{(l)} \in \left[z_k^{(l)-}, z_k^{(l)+}\right]$ can be bounded by

$$D_{N,H}(\bar{\omega}) \leq D_N(\hat{\omega}) + D_N(\omega)(1+2M)^s .$$

- Origin avoidance depends on how the new elements are generates inside $\left[z_k^{(l)-}, z_k^{(l)+}\right]$!

Effort for generation is only $\mathcal{O}(N \log N)$!

- Using pre-sorting of $H^{(1)}(\hat{\omega}) \Rightarrow$ Effort for generation can be lowered to $\mathcal{O}(N \log N)$:
  1. Generation of u.d. sequ. $(\hat{\omega}_N)$, (ω_N) ... effort $\mathcal{O}(N)$
  2. Calculation of $\hat{H}_N = H^{(1)}(\hat{\omega}_N)$ ... effort $\mathcal{O}(N)$
  3. Pre-sorting of support points $\hat{H}_N$ ... effort $\mathcal{O}(N \log N)$
  4. For each $1 \leq n \leq N$ and each dimension $1 \leq l \leq m$:
     - i Finding $\hat{z}_n^{(l)-}$ and $\hat{z}_n^{(l)+}$ ... effort $\mathcal{O}(N \log N)$
     - ii Calculation of result $y_k^{(l)}$ from them ... effort $\mathcal{O}(N)$
- Combined this gives in effort of $\mathcal{O}(N \log N)$ compared to $\mathcal{O}(N^2)$ for the Hlawka-Mück method

Thank you for your attention!